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T1 vs. T2: On the Definition of Mixed Strategies in Noncooperative Games

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T_1 vs. T_2: On the Definition of Mixed Strategies in Noncooperative Games

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Abstract. This paper identifies a central role of the topological separation axiom T_1 in the definition of mixed strategies in noncooperative games with arbitrary pure strategy spaces. Our main result says that the pure strategy space is T_1 if and only if the canonical mapping that assigns to each point its Dirac measure is injective and takes values in the space of regular Borel probability measures. In the absence of the T_1 separation axiom, pure strategies may not be available as mixed strategies and different pure strategies may correspond to the same mixed strategy. Moreover, there may be no mixed strategies with finite topological support. The analysis therefore suggests that the T_1 separation axiom is a natural minimum requirement on a pure strategy space when randomization is allowed for.

Keywords. Mixed strategies · Hausdorff spaces · T_1 separation axiom

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1 Introduction

Since the path-breaking contributions of Glicksberg (1952) and Fan (1953), it has been standard to define a mixed strategy as a regular probability measure on the Borel sets of the underlying pure strategy space.[1] Further, it has been common to assume that the topology on the underlying pure strategy spaces is Hausdorff. However, the rationale for using the Hausdorff separation axiom, except maybe that it naturally holds in metrizable spaces, has remained less clear. For example, it is known from Sion (1958) and Reny (1999) that neither the existence of a value in a two-person zero-sum game nor the existence of pure-strategy Nash equilibrium (PSNE) in a noncooperative game hinges on any separation axiom.[2] It is therefore remarkable that, with the sole exception of Mertens (1986), the Hausdorff assumption has been imposed in virtually every analysis of mixed-strategy Nash equilibrium (MSNE).

In this paper, we consider noncooperative games in which pure strategy spaces are arbitrary topological spaces, and explore the role of the T_1 separation axiom for the definition of mixed strategies. Imposed on a pure strategy set X equipped with some topology, the T_1 separation axiom says that, for any pure strategy $x \in X$, the singleton set $\{x\}$ is a closed set. The T_1 separation axiom is necessarily satisfied in all Hausdorff (i.e., T_2) strategy spaces, but conversely, T_1-spaces need not be Hausdorff. Our key result says that the topology on a player's pure strategy space satisfies the T_1 separation axiom if and only if the canonical mapping that assigns to each point in the space

[1]See, however, Kuhn (Kuhn, 1953) and Aumann (Aumann, 1964) for adaptations of the concept of mixed strategies in finite and infinite extensive games, respectively, and Milgrom and Weber (Milgrom and Weber, 1985) for an adaptation to Bayesian games.

[2]Similarly, Balder (1999, App. A) extended Kakutani's fixed point theorem to non-Hausdorff spaces. See also the more recent work by Goubault-Larrecq (2018) and Khan et al. (2024).

the corresponding Dirac measure is injective and all Dirac measures are regular. Thus, T_1 is equivalent to assuming that the space of pure strategies is canonically represented as a subset of the space of mixed strategies.

In the absence of the T_1 assumption, several intuitions about mixed strategies break down. Our first example illustrates the crucial role of topological separation properties for the definition of mixed strategies. For convenience, the example works with a finite normal-form game, but the discussion captures a more general problem.

Example 1. *Consider a Prisoner's Dilemma game and suppose that the pure strategy spaces, $X = Y = \{C, D\}$, are equipped with the* Sierpiński topology $\mathcal{T} = \{\varnothing, \{D\}, \{C, D\}\}$. *Let μ be any mixed strategy. The set $\{C\}$ is closed, hence Borel. By regularity, for any $\varepsilon \in (0, 1)$, there exists a closed set $K \subseteq \{C\}$ and an open set $U \supseteq \{C\}$ such that $\mu(U \setminus K) < \varepsilon$.*[3] *But the only open set containing C is $X = \{C, D\}$, so $U = X$. If $K = \varnothing$, then $\mu(U \setminus K) = 1$, which is impossible. Therefore $K = \{C\}$, and hence, $\mu(\{D\}) < \varepsilon$. Thus, $\mu(\{D\}) = 0$, and each player has only one mixed strategy, namely δ_C. Therefore, the unique MSNE leads to the pure outcome (C, C).*

Thus, in the absence of the T_1 separation axiom, one can define a topology on the pure strategy spaces of a Prisoner's Dilemma game such that cooperation becomes the unique MSNE outcome. Because the Dirac measure associated with the pure strategy D is not regular, it is not available as a degenerate mixed strategy, which changes the nature of the game in an undesirable way.

Indeed, if the pure strategy space does not satisfy the T_1 separation axiom, further unexpected things may happen. We will provide an example of a strategy space that is not T_1 and for which different pure strategies correspond to the same mixed strategy. In fact, in this example, all finite

[3] Cf. Section 2.

convex mixtures of Dirac measures coincide. Moreover, there are no mixed strategies with finite topological support. Such pathologies must therefore be kept in mind when using standard terminology such as "finitely supported strategies" in settings that do not impose the T_1 separation axiom.

The remainder of this paper is organized as follows. Section 2 concerns preliminaries. Section 3 studies the relationship between pure and mixed strategies. Section 4 discusses implications for mixed strategies with finite topological support. Section 5 concludes.

2 Preliminaries

This section prepares the analysis by reviewing the necessary background on separation axioms (Subsection 2.1) and regular probability measures (Subsection 2.2).

2.1 Separation Axioms in General Topology

A topological space[4] $(X, \mathcal{T})$ is called a T_0-space (or satisfies the T_0 *separation axiom*) if, for any two different points $x, x' \in X$, there exists $U \in \mathcal{T}$ such that either (i) $x \in U$ and $x' \notin U$, or (ii) $x' \in U$ and $x \notin U$ (Kelley, 1975, p. 57).

The space X is a T_1-space (or satisfies the T_1 *separation axiom*) if any singleton is closed (Kelley, 1975, p. 56). The following lemma provides an alternative way to think about the T_1 separation axiom.[5]

[4] A *topology* $\mathcal{T}$ on X is a collection of *open* subsets of X such that (i) $\varnothing$ and X are open, (ii) the union of any collection of open sets is open, and (iii) the intersection of any finite collection of open sets is open. A *topological space* $(X, \mathcal{T})$ consists of a set X and a topology $\mathcal{T}$ on X. A set is *closed* in X if its complement is open.

[5] For further discussion of T_1-spaces, see Reilly (1992) and Clontz (2026).

Lemma 1. *A topological space X satisfies the T_1 separation axiom if and only if, for any pair of different points $x, x' \in X$, there exists an open set U such that $x \in U$ and $x' \notin U$.*

Proof. See Engelking (1977, Sec. 1.5). $\square$

A topological space X is *Hausdorff* (is a T_2-*space*, or satisfies the T_2 *separation axiom*) if, for any pair of different points $x, x' \in X$, there are disjoint sets $U, U' \in \mathcal{T}$ such that $x \in U$ and $x' \in U'$. Any Hausdorff space is T_1, and any T_1-space satisfies T_0, but the corresponding reverse implications are not generally true.

The Hausdorff property is equivalent to the condition that the limit of any convergent net is unique (Kelley, 1975, Thm. 2.3). Moreover, in Hausdorff spaces, compact sets are closed. Conversely, however, the property that compact sets are closed is in general strictly weaker than the Hausdorff property even for a compact space (Wilansky, 1967, Thm. 1).

2.2 Regular Probability Measures

Given a measurable space (X, Σ),[6] a *probability measure* is a countably additive, nonnegative set function $\mu : \Sigma \to \mathbb{R}$ such that $\mu(X) = 1$. For $x \in X$, the *Dirac* measure δ_x is given by $\delta_x(S) = 0$ if $x \notin S$ and $\delta_x(S) = 1$ if $x \in S$, for any $S \in \Sigma$.

Given a topological space X, the elements of the smallest σ-algebra $\mathcal{B}(X)$ containing all open sets are called the *Borel sets*. Let $\widetilde{\mathcal{P}}(X)$ denote the set of

[6] A *σ-algebra* Σ is a collection of subsets of X such that (i) $X \in \Sigma$, (ii) if $A \in \Sigma$, then $A^c = X \setminus A \in \Sigma$, and (iii) if $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$. A *measurable space* is a pair (X, Σ), where X is a nonempty set and Σ is a σ-algebra on X. Given a measurable space (X, Σ), a *set function* is a mapping $\mu : \Sigma \to \mathbb{R}$. A set function is *nonnegative* if it does not attain negative values. A set function is *countably additive* if, for any countable collection $\{S_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in Σ, we have $\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n)$.

probability measures μ defined on the Borel sets of X. There is a canonical map that assigns to $x \in X$ the Dirac measure $\delta_x \in \widetilde{\mathcal{P}}(X)$.

A Borel probability measure $\mu \in \widetilde{\mathcal{P}}(X)$ is *regular* if for every $S \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is a closed set K and an open set U such that $K \subseteq S \subseteq U$ and $\mu(U \backslash K) < \varepsilon$ (Dunford and Schwartz, 1958, Sec. III.5). We write $\mathcal{P}(X)$ for the set of regular probability measures on X, and note that $\mathcal{P}(X) \subseteq \widetilde{\mathcal{P}}(X)$.

The *support* of a Borel probability measure $\mu \in \widetilde{\mathcal{P}}(X)$ is the intersection of all closed sets $S \subseteq X$ such that $\mu(S) = 1$.

3 Pure and Mixed Strategies

In noncooperative game theory, a *mixed strategy* is defined as a regular probability measure over a given topological space of pure strategies X. Thus, the space of mixed strategies is simply $\mathcal{P}(X)$. The definition of a mixed strategy does not require any separation axiom on X. The following result considers the implications of assuming T_0 only.

Proposition 1. *A pure strategy space X satisfies the T_0 separation axiom if and only if the canonical map that assigns to each $x \in X$ the Dirac measure $\delta_x \in \widetilde{\mathcal{P}}(X)$ is injective.*

Proof. *(Only if)* Let $x, x' \in X$ be such that $x \neq x'$. Because X is T_0, there exists an open set U such that either (i) $x \in U$ and $x' \notin U$, or (ii) $x' \in U$ and $x \notin U$. Then, U is Borel. In case (i), $\delta_x(U) = 1$, while $\delta_{x'}(U) = 0$. Similarly, in case (ii), $\delta_{x'}(U) = 1$, while $\delta_x(U) = 0$.

(If) Suppose that X is not T_0. Then, there are $x, x' \in X$ with $x \neq x'$ such that, for any open set U, we have that $x \in U$ is equivalent to $x' \in U$. Let $\hat{\Sigma} = \{S \subseteq X : x \in S \iff x' \in S\}$. Then, $\hat{\Sigma}$ is a σ-algebra containing

$\mathcal{B}(X)$. Hence, for any Borel set S, we have $x \in S$ if and only if $x' \in S$. But then, $\delta_x(S) = \delta_{x'}(S)$ for any $S \in \mathcal{B}(X)$, and the canonical mapping $x \mapsto \delta_x$ cannot be injective. $\square$

Example 3 will show that the canonical mapping $x \mapsto \delta_x$ indeed need not be injective in spaces that do not satisfy T_0.

The next result concerns the T_1 separation axiom.

Proposition 2. *A pure strategy space X satisfies the T_1 separation axiom if and only if the following two conditions are simultaneously satisfied:*

(i) X is a T_0-space;

(ii) any Dirac measure δ_x is regular.

Proof. *(Only if)* Property (i) follows from the usual hierarchy between T_1 and T_0. Next, we will show that the Dirac measure δ_x is regular. By definition, δ_x is regular if for each $S \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is a closed set $K \subseteq S$ and an open set $U \supseteq S$ such that $\delta_x(U \backslash K) < \varepsilon$. Take a Borel set $S \subseteq X$ and some $\varepsilon > 0$. There are two cases. Suppose first that $x \notin S$. Then, we choose the closed set $K = \varnothing \subseteq S$ and the open set $U = X \backslash \{x\} \supseteq S$ and note that $\delta_x(U \backslash K) = \delta_x(X \backslash \{x\}) = 0 < \varepsilon$. Suppose next that $x \in S$. Then, we choose the closed set $K = \{x\} \subseteq S$ and the open set $U = X \supseteq S$, noting that $\delta_x(U \backslash K) = \delta_x(X \backslash \{x\}) = 0 < \varepsilon$. In sum, we have shown that, indeed, δ_x is regular, so that property (ii) has been verified as well.

(If) Suppose that X is not T_1. Then, by Lemma 1, there exists a pair of distinct points $x, x' \in X$ such that any open set containing x contains also x'. Since X is T_0, there is an open set V such that $x' \in V$ and $x \notin V$. Let $S = X \setminus V$. Then S is closed, $x \in S$, and $x' \notin S$. By regularity, there exist a closed set $K \subseteq S$ and an open set $U \supseteq S$ such that $\delta_{x'}(U \setminus K) < 1$. Since

$x \in U$, we have $x' \in U$, whereas $K \subseteq S$ implies $x' \notin K$. Hence $x' \in U \setminus K$, so $\delta_{x'}(U \setminus K) = 1$, which yields the desired contradiction. □

In Example 1, which features T_0 pure-strategy spaces, the Dirac measure δ_D is irregular.

Taken together, the two propositions above have the following important implication.

Corollary 1. *A topological space X is T_1 if and only if the canonical mapping $x \mapsto \delta_x$ is injective and maps X into $\mathcal{P}(X)$.*

Proof. The claim is immediate from Propositions 1 and 2. □

The corollary shows that the T_1 separation axiom is a desirable condition on the topology of a pure strategy space. With the assumption in place, every pure strategy induces a regular Dirac measure and the map $x \to \delta_x$ is one-to-one. Thus, the T_1 separation axiom ensures that a player has access to her pure strategies in the mixed extension in the sense of a set-theoretic inclusion.

We conclude this section with an example illustrating Corollary 1.

Example 2. *An infinite set X equipped with the cofinite topology is compact and T_1 but not Hausdorff. By Corollary 1, Dirac measures corresponding to distinct points in X differ and every Dirac measure is regular. Thus, the pure-strategy space is indeed canonically represented as a subset of the mixed-strategy space.*

4 Finitely supported strategies

We have seen above that the T_1 separation axiom ensures that any Dirac measure is regular. As finite sums of regular measures are regular, it follows that, in a T_1-space, any convex combination of Dirac measures,

$$\mu = \sum_{m=1}^{M} \lambda_m \delta_{x_m},$$

with mass points $x_1, \ldots, x_M \in X$ and probability weights $\lambda_1, \ldots, \lambda_M \geq 0$ such that $\sum_{m=1}^{M} \lambda_m = 1$, is regular. Thus, the T_1 separation axiom ensures that all convex combinations of Dirac measures qualify as mixed strategies. Moreover, the support of any such strategy is a subset of $\{x_1, \ldots, x_M\}$ in that case because the T_1 separation axiom ensures that every finite set is closed. Therefore, in a T_1-space, any convex combination of Dirac measures is *finitely supported* in the topological sense, i.e., its support is a finite set.

In the absence of the T_1 separation axiom, however, this need not be true any longer. Thus, for a general pure strategy space, finite convex mixtures of Dirac measures need not be finitely supported. In fact, there may be no finitely supported probability measures at all, let alone finitely supported mixed strategies.

Example 3. *Let X be an infinite set equipped with the* indiscrete topology*, i.e., $\mathcal{T} = \{\varnothing, X\}$. Then, $\mathcal{B}(X) = \{\varnothing, X\}$, and there is a unique Borel probability measure μ. This measure is regular, and its support is X. In particular, $\delta_x = \mu$ for every $x \in X$. Consequently, the support of each Dirac measure is the infinite set X.*

Thus, there are no finitely supported mixed strategies in this case. Indeed, being the coarsest topology of all, the indiscrete topology in the example does not even satisfy the T_0 separation axiom.

5 Conclusion

While mixed strategies have been commonly defined over Hausdorff strategy spaces, the present analysis has shown that the more flexible T_1 separation axiom ("points are closed") may serve as a plausible alternative. As Corollary 1 shows, this axiom captures the requirement that a player, when allowed to randomize, does not lose access to any of her pure strategies.

We did not impose any restriction on the number of players. Therefore, all we have said applies equally to a Savage-style single-agent framework, where a decision maker chooses lotteries over acts. It also applies, in principle, to noncooperative games with infinitely many players.[7] Indeed, our comments concerned exclusively the relationship between pure and randomized choices or, as the case may be, between pure states and beliefs.

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[7] However, as kindly pointed out by the reviewer, infinite-player games raise additional issues, such as product measurability, product topology, and equilibrium existence, which are not studied in the present paper.

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